

Equilibrium properties of blackbody radiation in Doubly Special Relativity

Nitin Chandra^{*1}, Dheeraj Kumar Mishra^{†2}, and Vinay Vaibhav^{‡2}

¹B1(102), Om Sai Enclave, Hirabagh, Hazaribagh, 825301, India

²The Institute of Mathematical Sciences, Chennai, Tamil Nadu, 600113, India.

Abstract

Doubly Special Relativity (DSR) is an attempt to incorporate an observer independent energy/length scale in the relativistic theory. We study various thermodynamic properties of blackbody radiation in DSR. We find that the energy density, specific heat etc. follows usual acoustic phonon dynamics as have been well studied by Debye. Other thermodynamic quantities like pressure, entropy etc. have also been calculated. The usual Stefan-Boltzmann law gets modified. The phase-space measure is also expected to get modified for an exotic spacetime, which in turn leads to the modification of Planck energy density distribution and the Wien's displacement law.

1 Introduction

It seems that in all the theories attempting to combine Gravity with Quantum Mechanics, a natural length/energy scale emerges, i.e. Planck length/energy. This scale acts as a threshold where new description of spacetime is expected to appear. Doubly Special Relativity (DSR) attempts to incorporate this threshold as an invariant quantity under a relativistic transformation [1] [2]. This introduction of the observer independent energy scale (κ) leads to the modification in the dispersion relation of a free particle [1] [2] [3] [4]. The energy threshold also acts as a cut-off on the highest possible energy value in the physical (sub-Planckian) world.

It has been shown by Magueijo and Smolin (MS) that it is possible to keep the Lorentz group/algebra intact for the DSR theories. On the other hand the representation of the Lorentz group/algebra becomes non-linear to accomodate the invariant energy/length scale [3] [4]. Preserving the Lorentz group/algebra keeps the theory simple and intuitive. For our present study we will follow this DSR formulation by MS where the dispersion relation modifies

^{*}nitin.c.25@gmail.com

[†]dkmishra@imsc.res.in

[‡]vinayv@imsc.res.in

to,

$$\varepsilon^2 - p^2 = m^2 \left(1 - \frac{\varepsilon}{\kappa}\right)^2 \quad (1.1)$$

The Special Relativistic (SR) limit, i.e. $\kappa \rightarrow \infty$ gives the usual dispersion relation,

$$\varepsilon^2 - p^2 = m^2 \quad (1.2)$$

We will stick to the natural units ($\hbar = 1, c = 1, k_B = 1$) if not stated explicitly.

It is obvious that these changes (modification in dispersion relation and the presence of ultraviolet cut-off in energy) introduced will affect the thermodynamics of many well studied systems [5] [6] [7] [8] [9] [10]. In [9] an extensive study of classical ideal gas thermodynamics has been done using MS formalism. On the other hand [10] studies the photon gas thermodynamics in the same DSR formalism. It should be noted that their study contains serious flaws. They have considered the photon gas as a canonical ensemble obeying classical (Maxwell-Boltzmann) statistics. But it is a well known fact that photon gas follows a grand canonical ensemble (due to non-conservation of the photon number) and obeys the quantum (Bose-Einstein) statistics. Because of this severe error in their formalism the results obtained do not match with the usual photon gas thermodynamics (for example see section 7.3 of [11]). Surprisingly they match their results to the massless limit of the classical ideal gas thermodynamics in SR.

For a photon, the mass being zero, dispersion relation remains same as in the case of SR, i.e. $\varepsilon = p$. The DSR effect for the equilibrium properties of blackbody radiation is basically due to the cut-off κ in energy. We model the blackbody radiation in equilibrium as a grand canonical ensemble of photons obeying Bose-Einstein statistics as usually done¹. We have also considered the most general possible modification in the phase space measure for exotic spacetimes. In [9] a similar momentum dependent measure has been considered.

The present paper starts with the study of the possible changes due to the change in phase space measure. Next we go on calculating various thermodynamic quantities such as energy density, pressure, entropy etc. for both modified and unmodified measure and have compared and analyzed various results.

2 Change in phase-space measure for exotic spacetimes

Almost all the thermodynamic quantities for well studied systems encounter the large volume limit where discrete summation over energy values goes to the integration over phase space i.e.,

$$\sum_{\varepsilon} \rightarrow \frac{1}{(2\pi)^3} \int \int d^3x d^3p. \quad (2.1)$$

But for exotic spacetimes we expect the phase-space to modify as (see for example [9]),

$$\sum_{\varepsilon} \rightarrow \frac{1}{(2\pi)^3} \int \int d^3x d^3p f(\vec{x}, \vec{p}) \quad (2.2)$$

¹While the draft of this paper was being prepared we came to know about a very recent article by Mir Mehedi Faruk and Md. Muktadir Rahman [12] where they have also calculated the thermodynamic quantities of a photon gas for unmodified measure. It is interesting to note that our results differ significantly from theirs.

Here we have considered the most general possible modification. Assuming the spacetime to be isotropic and $f(\vec{x}, \vec{p}) = f(r, p)$ to be taylor series expandable in the powers of $(\frac{1}{r\kappa})$ and $(\frac{\varepsilon}{\kappa})$ we get,

$$f(r, p) = \sum_{n=0, n'=0}^{\infty} \frac{a_{n, n'}}{n! n'!} \left(\frac{\varepsilon}{\kappa}\right)^n \left(\frac{1}{r\kappa}\right)^{n'}, \quad (2.3)$$

with $a_{0,0} = 1$ as for $\kappa \rightarrow \infty$ we expect $f(r, p) \rightarrow 1$. This expansion is valid only when $\frac{\varepsilon}{\kappa}, \frac{1}{r\kappa} < 1$ throughout the integration range this requires $\varepsilon < \kappa$ and $r > \frac{1}{\kappa}$. Thus κ acts as highest energy cut-off while $\frac{1}{\kappa}$ acts as the lowest length cut-off. Due to this change in measure the integral of the form $\frac{1}{(2\pi)^3} \int \int d^3x d^3p F(\varepsilon)$ changes to

$$\begin{aligned} \frac{1}{(2\pi)^3} \int \int d^3x d^3p f(r, p) F(\varepsilon) &= \frac{1}{(2\pi)^3} \int_{r=\frac{1}{\kappa}}^R \int_{p=0}^{\kappa} d^3x d^3p \sum_{n=0, n'=0}^{\infty} \frac{a_{n, n'}}{n! n'!} \left(\frac{\varepsilon}{\kappa}\right)^n \left(\frac{1}{r\kappa}\right)^{n'} F(\varepsilon) \\ &= \frac{1}{(2\pi)^3} \sum_{n=0, n'=0}^{\infty} \frac{a_{n, n'}}{n! n'! \kappa^{n+n'}} \int_{r=\frac{1}{\kappa}}^R \int_{p=0}^{\kappa} d^3x d^3p \varepsilon^n \left(\frac{1}{r}\right)^{n'} F(\varepsilon), \end{aligned} \quad (2.4)$$

R being the radius of the spherical volume considered. Here we have interchanged the double summation and the integration which is allowed if (see appendix A),

$$\sum_{n=0, n'=0}^{\infty} \frac{|a_{n, n'}|}{n! n'! \kappa^{n+n'}} \int_{r=\frac{1}{\kappa}}^R \int_{p=0}^{\kappa} d^3x d^3p \varepsilon^n \left(\frac{1}{r}\right)^{n'} |F(\varepsilon)| < \infty. \quad (2.5)$$

Performing the integration over the coordinate space we obtain

$$\begin{aligned} \frac{1}{(2\pi)^3} \int \int d^3x d^3p f(r, p) F(\varepsilon) &= \frac{1}{(2\pi)^3} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n, n'}}{n! n'! \kappa^{n+3}} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi}\right)^{\frac{3-n'}{3}} - 1 \right] \int_{p=0}^{\kappa} d^3p \varepsilon^n F(\varepsilon) \\ &+ \frac{1}{(2\pi)^3} \sum_{n=0}^{\infty} \frac{a_{n, 3}}{n! 3! \kappa^{n+3}} \left(\frac{4\pi}{3}\right) \ln\left(\frac{3V\kappa^3}{4\pi}\right) \int_{p=0}^{\kappa} d^3p \varepsilon^n F(\varepsilon) \end{aligned} \quad (2.6)$$

where $V = \frac{4}{3}\pi R^3$ is the volume of the spherical ball of radius R . The accessible part of the volume for the particle is $V_{ac} = V - \frac{4\pi}{3\kappa^3}$. For the large volume limit the minimum length $\frac{1}{\kappa} \ll R$ implying $V\kappa^3 \gg 1$ which in turn implies $V_{ac} \approx V$. Note that a small volume $\frac{4\pi}{3\kappa^3}$ is inaccessible to each particle. This inaccessible volume can be extracted out at any point in the space as volume being large all the space points are equivalent. We have extracted out this volume at the center $r = 0$.

Example: Classical Ideal gas in canonical ensemble

Let us take a particular example of $F(\varepsilon)$ to illustrate this further. We consider the classical ideal gas in canonical ensemble obeying Maxwell-Boltzmann statistics with the partition function [11],

$$Z_N(V_{ac}, T) = \sum_E \exp[-\beta E] = \frac{1}{N!} [Z_1(V_{ac}, T)]^N, \quad (2.7)$$

where $Z_1(V_{ac}, T)$ is the single particle partition function, N is the total number of constituent particles, $\beta = \frac{1}{T}$ and the total energy E of the system is,

$$E = \sum_{\varepsilon} n_{\varepsilon} \varepsilon. \quad (2.8)$$

Here n_ε is the number of particles corresponding to the single particle energy ε . Obviously,

$$\sum_{\varepsilon} n_{\varepsilon} = N. \quad (2.9)$$

The single particle partition function $Z_1(V_{ac}, T)$ is given by,

$$Z_1(V_{ac}, T) = \sum_{\varepsilon} \exp[-\beta(\varepsilon - m_0)]. \quad (2.10)$$

Taking the large volume limit (2.2) we obtain,

$$Z_1(V_{ac}, T) = \frac{1}{(2\pi)^3} \int \int d^3x d^3p f(r, p) \exp(-\beta(\varepsilon - m_0)). \quad (2.11)$$

Using (2.6) for $F(\varepsilon) = \exp(-\beta(\varepsilon - m_0))$ and following the arguments given in [9] we get

$$\begin{aligned} Z_1(V_{ac}, T) &= \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'\kappa^n} \left(\frac{3}{(3-n')(\kappa^3 V_{ac})} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left(m_0 - \frac{\partial}{\partial \beta} \right)^n Z_1^0(V_{ac}, T) \\ &+ \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!\kappa^n} \left(\frac{4\pi}{18\kappa^3 V_{ac}} \right) \ln \left(\frac{3V\kappa^3}{4\pi} \right) \left(m_0 - \frac{\partial}{\partial \beta} \right)^n Z_1^0(V_{ac}, T) \end{aligned} \quad (2.12)$$

where $Z_1^0(V_{ac}, T)$ is the single particle partition function with unmodified measure,

$$Z_1^0(V_{ac}, T) = \frac{V_{ac}}{(2\pi)^3} \int_{p=0}^{\kappa} d^3p \exp(-\beta(\varepsilon - m_0)). \quad (2.13)$$

The expression of $Z_1(V_{ac}, T)$ has now non-trivial dependence on V unlike in case of $Z_1^0(V_{ac}, T)$. With this modification the value of thermodynamic quantities, especially pressure, change. Let's not digress any more and continue with the study of the photon gas thermodynamics.

3 Equilibrium properties of blackbody radiation with unmodified measure

In this section we will see the possible changes in thermodynamic quantities of photon gas without considering the change in phase space measure. The model contains an ideal gas of identical and indistinguishable quanta namely, photons, [11]. There are n_ω number of photons each with energy $\varepsilon = \omega$. The mean value of n_ω is,

$$\langle n_\omega \rangle = \frac{\sum_{n_\omega=0}^{\infty} n_\omega e^{-\frac{n_\omega \omega}{T}}}{\sum_{n_\omega=0}^{\infty} e^{-\frac{n_\omega \omega}{T}}} = \frac{1}{e^{\frac{\omega}{T}} - 1} \quad (3.1)$$

giving mean energy as,

$$\langle \varepsilon \rangle = \omega \langle n_\omega \rangle = \frac{\omega}{e^{\frac{\omega}{T}} - 1} \quad (3.2)$$

In the large volume limit the volume of the phase space can be used to find the number of modes between the range ω and $\omega + d\omega$ which is given by,

$$a(\omega)d\omega = \frac{2}{(2\pi)^3} (4\pi p^2 dp) \int d^3x = \frac{V_{ac}\omega^2 d\omega}{\pi^2} \quad (3.3)$$

Note that photons obey the dispersion relation $\omega = \varepsilon = p$. The factor 2 comes due to the 2 tranverse polarizations of a photon. It is also to be noted that the above expression will get modified when we consider the change of the phase space measure. The energy density distribution therefore becomes,

$$u(\omega)d\omega = \frac{a(\omega)d\omega}{V_{ac}} \langle \varepsilon \rangle = \frac{1}{\pi^2} \frac{\omega^3 d\omega}{e^{\frac{\omega}{T}} - 1} \quad (3.4)$$

This is the usual Planck energy density distribution.

3.1 Energy Density

Integrating 3.4 from $\omega = 0$ to $\omega = \kappa$ we get the energy density of the photon gas as,

$$u \equiv \frac{U}{V_{ac}} = \int_0^\kappa u(\omega)d\omega = \frac{T^4}{\pi^2} \int_0^{\frac{\kappa}{T}} \frac{x^3 dx}{e^x - 1} = \frac{6T^4}{\pi^2} \left[Z_4(0) - Z_4\left(\frac{\kappa}{T}\right) \right] \quad (3.5)$$

Here we have changed the variable to $x = \frac{\omega}{T}$. Note that at finite and non zero T , as $\kappa \rightarrow \infty$ this expression reduces to the one given in equation (12) on page 203 of [11], giving usual law. Also $Z_n(x)$ is the incomplete zeta functions or “Debye functions”(refer to section 27.1 of [16]), and is given as,

$$Z_n(x) = \frac{1}{\Gamma(n)} \int_x^\infty \frac{t^{n-1}}{e^t - 1} dt \quad (3.6)$$

We note that $Z_n(0) = \zeta(n)$ where $\zeta(z)$ is the Riemann-Zeta function, in particular $Z_4(0) = \zeta(4) = \frac{\pi^4}{90}$. It is remarkable that 3.5 is exactly same as in case of acoustic phonons ([17]) with the replacements $\kappa \rightarrow \Theta_D$ (Debye temperature), $2(\text{number of photon polarizations}) \rightarrow 3(\text{number of acoustic modes in monoatomic Bravais lattice})$ and the velocity of acoustic phonons has to be taken to be equal to 1 for correct matching as we are working in natural units. In case of acoustic phonons the cut-off on the possible frequencies comes due to the finiteness of first Brillouin zone which itself is restricted by the number density of ions in the lattice. On the other hand the energy cut-off κ in 3.5 comes from the quantum gravity considerations. We expect the specific heat $C_V = \left(\frac{\partial U}{\partial T}\right)_{V_{ac}} = T \left(\frac{\partial S}{\partial T}\right)_{V_{ac}}$ for a DSR photon gas to follow the behaviour of C_V as in case of acoustic phonons. For a mathematically rigorous treatment of Debye theory see [14]. Debye functions $Z_n(z)$ are related to the polylogarithm function $Li_n(z)$ by (see equation (16.2) of [15])

$$Z_n(z) = \sum_{k=0}^{n-1} Li_{n-k}(e^{-z}) \frac{z^k}{k!}. \quad (3.7)$$

for $n > 0$. Especially $Z_n(0) = Li_n(0)$. Here polylogarithm functions themselves can be series expanded for $|z| < 1$ as (see equation (8.1) of [15])

$$Li_n(z) = \sum_{a=1}^{\infty} \frac{z^a}{a^n}. \quad (3.8)$$

The integral representation of $Li_n(z)$ is, for $Re(n) > 0$, as follows (see equation (1) of [15])

$$Li_n(z) = \frac{z}{\Gamma(n)} \int_0^\infty \frac{t^{n-1}}{e^t - z} dt \quad (3.9)$$

Thus the energy density can be written in terms of $Li_n(z)$ as given below

$$u = \frac{\pi^2 T^4}{15} - \left[\left(\frac{6T^4}{\pi^2} \right) Li_4(e^{-\frac{\kappa}{T}}) + \left(\frac{6\kappa T^3}{\pi^2} \right) Li_3(e^{-\frac{\kappa}{T}}) + \left(\frac{3\kappa^2 T^2}{\pi^2} \right) Li_2(e^{-\frac{\kappa}{T}}) - \left(\frac{\kappa^3 T}{\pi^2} \right) \ln(1 - e^{-\frac{\kappa}{T}}) \right] \quad (3.10)$$

Where we have used $Li_1(z) = -\ln(1 - z)$ (see equation (6.1) of [15]). Note that the first term corresponds to the usual Stefan-Boltzmann law. All the other terms modify the law which in turn may give a correction to the temperature measurements of different stellar objects. These correction terms vanish in the SR limit.

3.2 Specific heat

We put $\frac{U}{T} = u \frac{V_{ac}}{T}$ and use (3.10) along with using the derivatives of polylogarithm given by (see equation 4.1 of [15]),

$$\frac{\partial}{\partial \mu} [Li_n(e^\mu)] = Li_{n-1}(e^\mu). \quad (3.11)$$

and obtain the expression for specific heat as,

$$\begin{aligned} C_V &= \left(\frac{\partial U}{\partial T} \right)_{V_{ac}} \\ &= -\frac{\kappa^4 V_{ac}}{\pi^2 T} \frac{1}{(e^{\frac{\kappa}{T}} - 1)} + \left[\frac{4\pi^2 T^3 V_{ac}}{15} - \left(\frac{24T^3 V_{ac}}{\pi^2} \right) Li_4(e^{-\frac{\kappa}{T}}) - \left(\frac{24\kappa T^2 V_{ac}}{\pi^2} \right) Li_3(e^{-\frac{\kappa}{T}}) \right. \\ &\quad \left. - \left(\frac{12\kappa^2 T V_{ac}}{\pi^2} \right) Li_2(e^{-\frac{\kappa}{T}}) + \frac{4V_{ac}\kappa^3}{\pi^2} \ln(1 - e^{-\frac{\kappa}{T}}) \right] \end{aligned} \quad (3.12)$$

We now use the relationship between the polylogarithm functions and Debye functions given in (3.7) and the expression for energy density given in (3.5) to obtain,

$$C_V = -\frac{\kappa^4 V_{ac}}{\pi^2 T} \frac{1}{(e^{\frac{\kappa}{T}} - 1)} + 4 \left(\frac{U}{T} \right) \quad (3.13)$$

Differentiating (3.5) one can easily show that,

$$C_V = \left(\frac{T}{\kappa} \right)^3 \frac{\kappa^3 V_{ac}}{\pi^2} \int_0^{\frac{\kappa}{T}} \frac{x^4 e^x dx}{(e^x - 1)^2} \quad (3.14)$$

This expression is exactly what we obtain in case of acoustic phonons (see equation (17), (18), (19) etc. of section 7.4 in [11]) subjected to the required replacements as discussed before. Again the SR limit gives the usual result given in equation (20) of section 7.3 of [11].

3.3 Radiation Pressure

The grand canonical partition function for the photon gas (with fugacity $z = 1$) is [11],

$$Q(V_{ac}, T) = \prod_{\epsilon} \frac{1}{1 - e^{-\frac{\epsilon}{T}}} \quad (3.15)$$

The q -potential then becomes,

$$q \equiv \frac{PV_{ac}}{T} \equiv \ln Q(V_{ac}, T) = - \sum_{\varepsilon} \ln(1 - e^{-\frac{\varepsilon}{T}}) \quad (3.16)$$

Again taking the large volume limit we obtain,

$$\begin{aligned} \left(\frac{PV_{ac}}{T} \right) &= \frac{-V_{ac}}{\pi^2} \int_0^{\kappa} \ln(1 - e^{-\frac{\varepsilon}{T}}) \varepsilon^2 d\varepsilon \\ &= \left[\frac{-V_{ac}}{\pi^2} \ln(1 - e^{-\frac{\varepsilon}{T}}) \frac{\varepsilon^3}{3} \right]_0^{\kappa} + \frac{V_{ac}}{3\pi^2 T} \int_0^{\kappa} \frac{\varepsilon^3 d\varepsilon}{e^{\frac{\varepsilon}{T}} - 1} \\ &= -\frac{\kappa^3 V_{ac}}{3\pi^2} \ln(1 - e^{-\frac{\kappa}{T}}) + \frac{T^3 V_{ac}}{3\pi^2} \int_0^{\frac{\kappa}{T}} \frac{x^3 dx}{e^x - 1} \end{aligned} \quad (3.17)$$

Using 3.5 we find,

$$P = -\frac{T\kappa^3}{3\pi^2} \ln(1 - e^{-\frac{\kappa}{T}}) + \frac{1}{3}u \quad (3.18)$$

Thus the equation of state for the blackbody radiation field, i.e., the relation between the pressure and the energy density, got modified and goes to the correct SR limit

$$P_{SR} = \frac{1}{3}(u)_{SR} = \frac{\pi^2 (T_{SR})^4}{45} \quad (3.19)$$

3.4 Entropy

The Helmholtz free energy is given by (the chemical potential $\mu = 0$),

$$A = \mu N - PV_{ac} = -PV_{ac} = \frac{T\kappa^3 V_{ac}}{3\pi^2} \ln(1 - e^{-\frac{\kappa}{T}}) - \left(\frac{U}{3} \right). \quad (3.20)$$

The entropy becomes,

$$S = \frac{U - A}{T} = -\frac{\kappa^3 V_{ac}}{3\pi^2} \ln(1 - e^{-\frac{\kappa}{T}}) + \frac{4}{3} \left(\frac{U}{T} \right) \quad (3.21)$$

In SR limit, the first term vanishes and the expression goes to the correct result (see equation (19) of section 7.3 in [11])

$$S_{SR} = \frac{4}{3} \left(\frac{U}{T} \right)_{SR}. \quad (3.22)$$

3.5 Equilibrium number of photons

The equilibrium number of photons can be obtained by integrating the product of mean number of photons (3.1) and the volume of the phase space (3.3),

$$\bar{N} = \int_0^{\kappa} \frac{V_{ac}}{\pi^2} \frac{\omega^2 d\omega}{e^{\frac{\omega}{T}} - 1} = \frac{2V_{ac}T^3}{\pi^2} \left[Z_3(0) - Z_3\left(\frac{\kappa}{T}\right) \right] \quad (3.23)$$

Here $Z_3(0) = \zeta(3)$ is also called Apery's constant. This with the proper replacements corresponds to the equilibrium number of acoustic phonons in Debye theory. In the $\kappa \rightarrow \infty$ limit $Z_n(\frac{\kappa}{T}) \rightarrow 0$ and $V_{ac} \rightarrow V$ and we get the usual SR result as given in equation (23) of section 7.3 of [11],

$$(\bar{N})_{SR} = \frac{T^3 V}{\pi^2} (2\zeta(3)) \quad (3.24)$$

4 Equilibrium properties with modified measure

We will now consider the change in phase space as described in section 2. This leads to the modification of the energy density distribution as well as the q potential which in effect modifies all the thermodynamic quantities.

4.1 Modified Planck's energy distribution and Wien's law

It is clear that the change in phase space is going to modify the Planck distribution for the energy density of the blackbody radiation. In such a scenario (3.3) modifies to,

$$\begin{aligned} a(\omega)d\omega &= \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^{n+3}} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \omega^{n+2} d\omega \\ &+ \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^{n+3}} \left(\frac{4\pi}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \omega^{n+2} d\omega \end{aligned} \quad (4.1)$$

Now the Planck energy density distribution (3.4) changes to,

$$\begin{aligned} u(\omega)d\omega &= \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{1}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \frac{\omega^{n+3} d\omega}{e^{\frac{\omega}{T}} - 1} \\ &+ \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi}{3\kappa^3 V_{ac}} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \frac{\omega^{n+3} d\omega}{e^{\frac{\omega}{T}} - 1} \end{aligned} \quad (4.2)$$

A typical plot of the modified energy density distribution in comparison to the usual Planck distribution is shown in Figure 1 on page 10. Let us first express the above distribution in terms of wavelength λ . The energy density between ω and $\omega + d\omega$ or the corresponding λ and $\lambda + d\lambda$ is given

$$u(\lambda)d\lambda = u(\omega)d\omega \quad (4.3)$$

which implies

$$u(\lambda) = u(\omega) \frac{d\omega}{d\lambda} = -\frac{\omega^2 u(\omega)}{2\pi} \quad (4.4)$$

ω and λ are related by $\omega = \frac{2\pi}{\lambda}$. From (4.2) we can write

$$u(\omega) = \sum_{n=0}^{\infty} A_n \frac{\omega^{n+3}}{e^{\frac{\omega}{T}} - 1} \quad (4.5)$$

where

$$\begin{aligned} A_n &= \frac{1}{\pi^2} \sum_{n'=0, n' \neq 3}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n} \left(\frac{4\pi}{3-n'} \right) \frac{1}{\kappa^3 V_{ac}} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \\ &+ \frac{1}{\pi^2} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi}{3\kappa^3 V_{ac}} \right) \ln \left(\frac{3V\kappa^3}{4\pi} \right) \end{aligned} \quad (4.6)$$

is a constant and is independent of both λ and T . We then have,

$$u(\lambda) = - \sum_{n=0}^{\infty} \frac{(2\pi)^{n+4} A_n}{\lambda^{n+5} \left(e^{\frac{2\pi}{\lambda T}} - 1 \right)} \quad (4.7)$$

Differentiating with respect to λ we get

$$\frac{du(\lambda)}{d\lambda} = \frac{1}{\lambda^6 \left(e^{\frac{2\pi}{\lambda T}} - 1 \right)} \sum_{n=0}^{\infty} \frac{(2\pi)^{n+4} A_n}{\lambda^n} \left[n + 5 - \frac{\left(\frac{2\pi}{\lambda T} \right)}{1 - e^{-\frac{2\pi}{\lambda T}}} \right] \quad (4.8)$$

Note that in case of unmodified measure, we have $A_0 = 1/\pi^2, A_1 = A_2 = \dots = 0$ and the above expression reduces to

$$\frac{du(\lambda)}{d\lambda} = \frac{1}{\lambda^6 \left(e^{\frac{2\pi}{\lambda T}} - 1 \right)} \frac{(2\pi)^4}{\pi^2} \left[5 - \frac{\left(\frac{2\pi}{\lambda T} \right)}{1 - e^{-\frac{2\pi}{\lambda T}}} \right] \quad (4.9)$$

$u(\lambda)$ is maximum at $\lambda = \lambda_{max}$ which can be found by the extremum condition $\left. \frac{du(\lambda)}{d\lambda} \right|_{\lambda_{max}} = 0$ giving,

$$5 = \frac{x_{max}}{1 - e^{-x_{max}}}, \quad (4.10)$$

where $x_{max} = \frac{2\pi}{\lambda_{max} T}$. The above equation can be numerically solved to get

$$x_{max} = \frac{2\pi}{\lambda_{max} T} \approx 4.965, \Rightarrow \lambda_{max} T \approx 1.266 \quad (4.11)$$

This behaviour of λ_{max} on temperature T is called *Wien's displacement law*. Now for the case of modified measure the extremum condition becomes,

$$\sum_{n=0}^{\infty} T^n x_{max}^n A_n \left[n + 5 - \frac{x_{max}}{1 - e^{-x_{max}}} \right] = 0 \quad (4.12)$$

It is obvious that the solution of x_{max} is now dependent on T . Thus the value of $x_{max} = \frac{2\pi}{\lambda_{max} T}$ is no more constant, but a function of T . To understand the behaviour in a better way, we keep the leading order terms in $\frac{T}{\kappa}$ and $\frac{1}{V^{1/3}\kappa}$ and neglect all the higher order terms i.e.,

$$A_0 \approx 1/\pi^2, \quad A_1 \approx \frac{a_{1,0}}{\pi^2 \kappa}, \quad A_2 \approx A_3 \approx \dots \approx 0 \quad (4.13)$$

The extremum condition then becomes

$$\frac{T}{\kappa} = - \frac{1}{x_{max} a_{1,0}} \left(\frac{5 - \frac{x_{max}}{1 - e^{-x_{max}}}}{6 - \frac{x_{max}}{1 - e^{-x_{max}}}} \right) = f(x_{max}) \quad (4.14)$$

We have plotted this function with respect to x_{max} (see Figure 2 on page 10). For a fixed value of y -axis, i.e., a fixed $\frac{T}{\kappa}$ value the corresponding value of x_{max} can be obtained from the plot. As visible from the plot $x_{max} = \frac{2\pi}{\lambda_{max} T}$ is a monotonically increasing function of T , i.e., $f^{-1}(\frac{T}{\kappa})$. This implies $\lambda_{max} = \frac{2\pi}{T f^{-1}(\frac{T}{\kappa})}$ is a monotonically decreasing function of T . Note that λ_{max} for modified phase space measure decreases more rapidly with increasing T than the case of unmodified measure where $x_{max} = f^{-1}(\frac{T}{\kappa})$ takes a constant value. The significant change in the values of x_{max} occurs only if the order of the change in temperature is non-negligible with respect to κ . That is why in SR limit, i.e., $\frac{T}{\kappa} \rightarrow 0$, the $x_{max} = \frac{2\pi}{\lambda_{max} T}$ is almost constant giving the standard Wien's displacement law. The extremum condition for the unmodified measure, i.e., (4.10) corresponds to $f(x_{max}) = 0$. As it is visible in Figure 2 this gives the usual value $x_{max} \approx 4.965$.

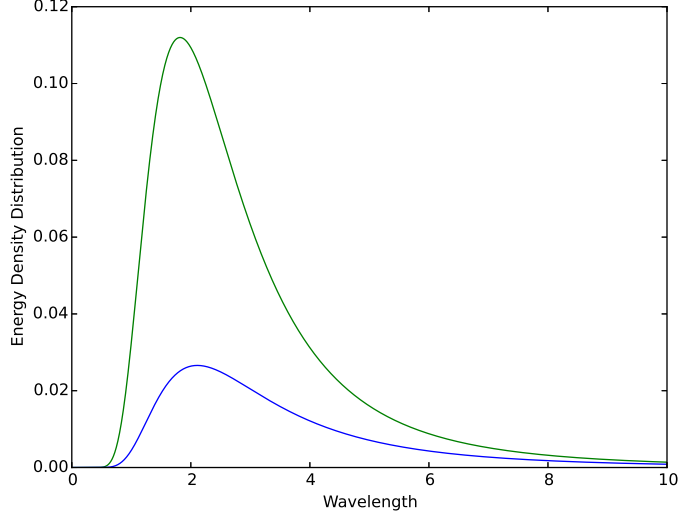


Figure 1: The plot showing the Planck energy distribution as a function of wavelength λ for both modified and unmodified measure. Here the blue and the green color correspond to the unmodified and the modified measure respectively. We have taken $a_{0,0} = 1.0$, $a_{0,1} = a_{1,0} = 0.2$ and all other a 's are zero, temperature is $T = 0.6$, volume is 10^{35} and $\kappa = 1$ in Planck units.

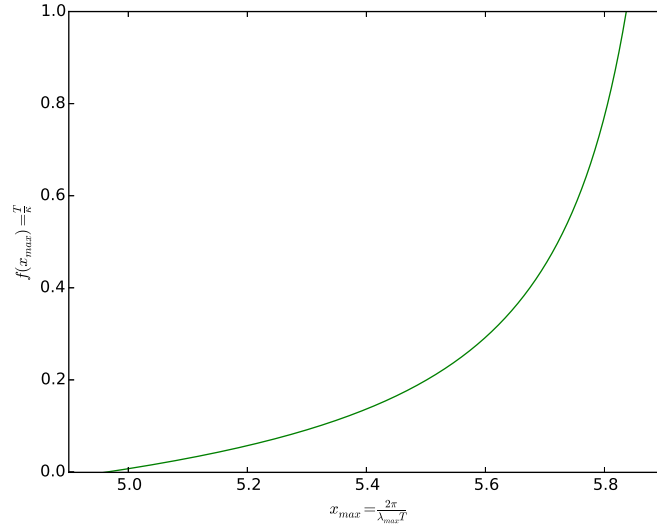


Figure 2: The plot showing the modification in the Wien's displacement law. The usual Wien's law would have given us a value of x_{max} corresponding to the point where $f(x_{max}) = 0$. We can clearly see from the plot that this value is $x_{max} \approx 4.965$. The coefficient $a_{1,0}$ has been taken as 1. We have chosen the range such that $\frac{T}{\kappa}$ lies between 0 and 1.

4.2 Energy Density

Following what we did in section 3.1 along with using (4.2) we get,

$$\begin{aligned}
u &= \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{T^{n+4}}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+4) \left[Z_{n+4}(0) - Z_{n+4}\left(\frac{\kappa}{T}\right) \right] \\
&+ \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi T^{n+4}}{3V_{ac}\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+4) \left[Z_{n+4}(0) - Z_{n+4}\left(\frac{\kappa}{T}\right) \right] \\
&= \sum_{n=0, n'=0}^{\infty} u_{n,n'}
\end{aligned} \tag{4.15}$$

Here $u_{n,n'}$ is a general term of the summation. This gives the Stefan-Boltzmann law for blackbody radiation in DSR with modified phase space measure.

4.3 Specific heat

Therefore the specific heat capacity C_V is,

$$\begin{aligned}
C_V &= \left(\frac{\partial U}{\partial T} \right)_{V_{ac}} = \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,n'}}{n!n'!\kappa^{n+3}} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left\{ \frac{1}{(1-e^{-\frac{\kappa}{T}})} \frac{\kappa^{n+4}}{T} \right\} + (n+4) \left(\frac{u_{n,n'} V_{ac}}{T} \right) \right] \\
&+ \sum_{n=0}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,3}}{n!3!\kappa^{n+3}} \left(\frac{4\pi}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \left\{ \frac{1}{(1-e^{-\frac{\kappa}{T}})} \frac{\kappa^{n+4}}{T} \right\} + (n+4) \left(\frac{u_{n,3} V_{ac}}{T} \right) \right]
\end{aligned} \tag{4.16}$$

4.4 Radiation Pressure

The radiation pressure can easily be calculated in essentially the similar manner to get,

$$\begin{aligned}
P &= \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{T}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left\{ -\ln(1-e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right. \\
&+ \left. \frac{T^{n+3}}{(n+3)} \Gamma(n+4) \left[Z_{n+4}(0) - Z_{n+4}\left(\frac{\kappa}{T}\right) \right] \right\} \\
&+ \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi T}{3V_{ac}\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \left\{ -\ln(1-e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} + \frac{T^{n+3}}{(n+3)} \Gamma(n+4) \left[Z_{n+4}(0) - Z_{n+4}\left(\frac{\kappa}{T}\right) \right] \right\}
\end{aligned} \tag{4.17}$$

This can be related to the energy density as

$$\begin{aligned}
P &= \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{T}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left\{ -\ln(1-e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} + \frac{u_{n,n'}}{(n+3)} \right] \\
&+ \sum_{n=0}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi T}{3V_{ac}\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \left\{ -\ln(1-e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} + \frac{u_{n,3}}{(n+3)} \right]
\end{aligned} \tag{4.18}$$

4.5 Entropy

Let us first calculate the Helmholtz free energy,

$$\begin{aligned}
A &= \mu N - PV_{ac} = -PV_{ac} \\
&= \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,n'}}{n!n'!\kappa^{n+3}} \frac{4\pi T}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left\{ \ln(1 - e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} - \frac{u_{n,n'} V_{ac}}{(n+3)} \right] \\
&\quad + \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^{n+3}} \left[\frac{1}{(\pi)^2} \left(\frac{4\pi T}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \left\{ \ln(1 - e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} - \frac{u_{n,3} V_{ac}}{(n+3)} \right]
\end{aligned} \tag{4.19}$$

And the entropy becomes,

$$\begin{aligned}
S = \frac{U - A}{T} &= \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,n'}}{n!n'!\kappa^{n+3}} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left\{ -\ln(1 - e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} + \frac{(n+4)}{(n+3)} \left(\frac{u_{n,n'} V_{ac}}{T} \right) \right] \\
&\quad + \sum_{n=0}^{\infty} \left[\frac{1}{(\pi)^2} \frac{a_{n,3}}{n!3!\kappa^{n+3}} \left(\frac{4\pi}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \left\{ -\ln(1 - e^{-\frac{\kappa}{T}}) \frac{\kappa^{n+3}}{n+3} \right\} + \frac{(n+4)}{(n+3)} \left(\frac{u_{n,3} V_{ac}}{T} \right) \right]
\end{aligned} \tag{4.20}$$

4.6 Equilibrium number of photons

The equilibrium number of photons in modified measure can be estimated in the same way as we did in unmodified case and using (4.1) we have,

$$\begin{aligned}
\bar{N} &= \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^{n+3}} \frac{4\pi T^{n+3}}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+3) \left[Z_{n+3}(0) - Z_{n+3} \left(\frac{\kappa}{T} \right) \right] \\
&\quad + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^{n+3}} \left(\frac{4\pi T^{n+3}}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+3) \left[Z_{n+3}(0) - Z_{n+3} \left(\frac{\kappa}{T} \right) \right]
\end{aligned} \tag{4.21}$$

In all the above thermodynamic quantities for modified measure taking the SR limit gives the usual result.

5 The leading behaviour for $T \rightarrow \kappa$ and $T \rightarrow 0$

We have plotted various thermodynamic quantities as a function of temperature (see Figure 3 on page 13). Let's now analyse the behaviour near $T = \kappa$ and $T = 0$. For high temperature $T \approx \kappa(1 - \epsilon)$ such that $\epsilon \ll 1$ which gives $\frac{\kappa}{T} \approx 1 + \epsilon$. We will expand all the quantities to the leading order in ϵ and finally put $\epsilon = 1 - \frac{T}{\kappa}$ to get the leading temperature behaviour. For unmodified measure we get

$$u \approx -\frac{18\kappa^4}{\pi^2} \left[Z_4(0) - Z_4(1) - \frac{1}{18(e-1)} \right] + \frac{24\kappa^4}{\pi^2} \left(\frac{T}{\kappa} \right) \left[Z_4(0) - Z_4(1) - \frac{1}{24(e-1)} \right] \tag{5.1}$$

$$P \approx -\frac{6\kappa^4}{\pi^2} [Z_4(0) - Z_4(1)] + \frac{8\kappa^4}{\pi^2} \left(\frac{T}{\kappa} \right) [Z_4(0) - Z_4(1)] - \frac{\kappa^4}{3\pi^2} \left(\frac{T}{\kappa} \right) \ln \left(1 - \frac{1}{e} \right) \tag{5.2}$$

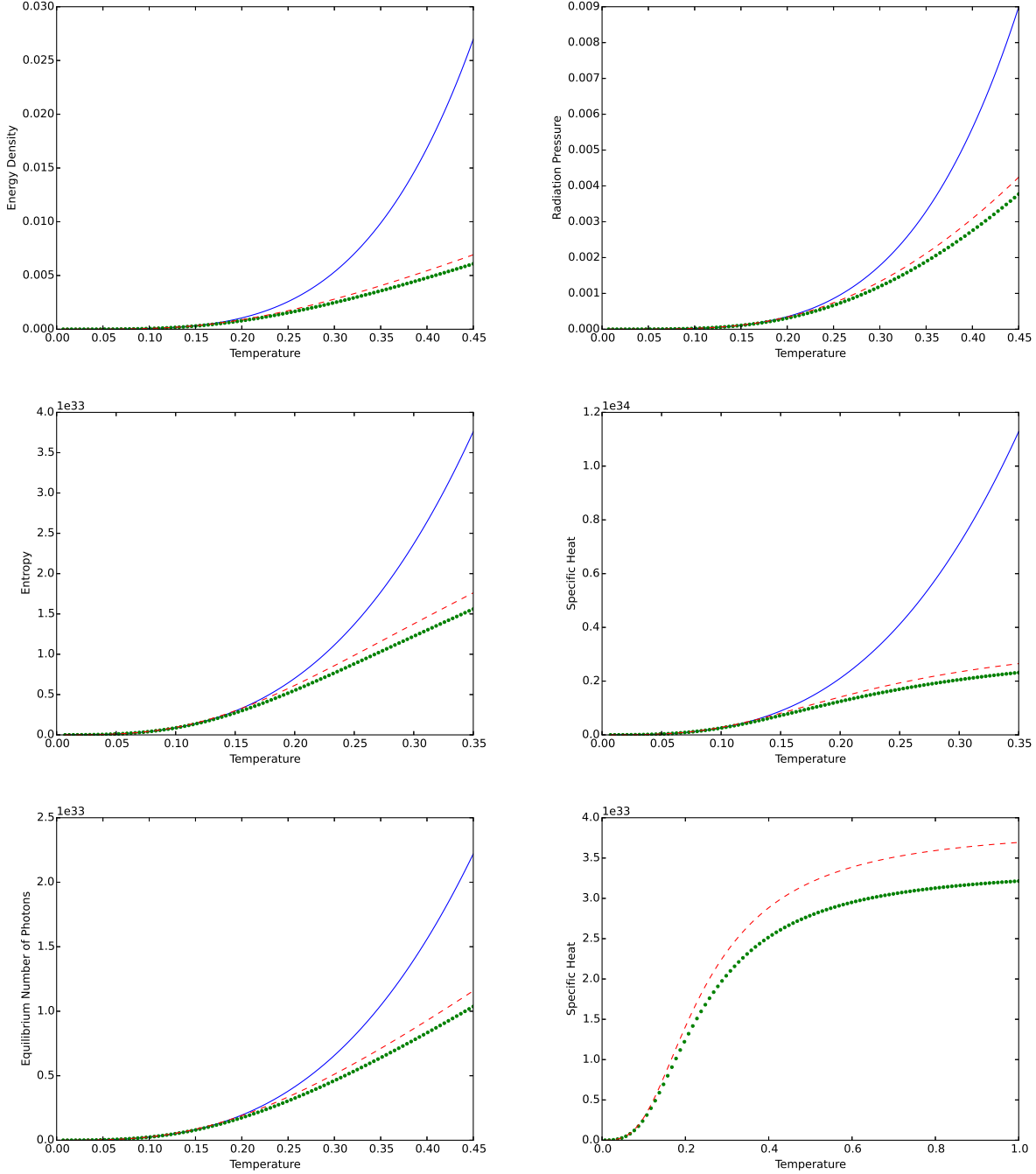


Figure 3: The plots show the variation of energy density, radiation pressure, entropy, specific heat and equilibrium number of photons with temperature for both unmodified and modified measure. The blue solid, the green dotted and the red dashed lines correspond to the SR, the DSR with unmodified measure and the DSR with modified measure respectively. As is visible from the plots it matches with SR at low T and with increasing T it deviates from SR significantly. Here $\kappa = 1$, $V = 10^{35}$ and $a_{0,0} = 1$, $a_{0,1} = a_{1,0} = 0.2$ and all other a 's are taken to be zero. Note that all the quantities above become approximately linear near $T \rightarrow \kappa$. The specific heat however goes as T^3 for lower T . The behaviour of C_V for the full range of $\frac{T}{\kappa} \in [0, 1]$ is shown in the figure at bottom right corner, which certainly mimics the Debye theory. In the Debye theory however T may go upto infinity in which case the specific heat goes to a constant value.

$$S \approx -\frac{\kappa^3 V_{ac}}{3\pi^2} \ln\left(1 - \frac{1}{e}\right) - \frac{16\kappa^3 V_{ac}}{\pi^2} \left[Z_4(0) - Z_4(1) - \frac{1}{48(e-1)} \right] + \frac{24\kappa^3 V_{ac}}{\pi^2} \left(\frac{T}{\kappa} \right) \left[Z_4(0) - Z_4(1) - \frac{1}{24(e-1)} \right] \quad (5.3)$$

$$C_V \approx -\frac{48\kappa^3 V_{ac}}{\pi^2} \left[Z_4(0) - Z_4(1) - \frac{(3e-2)}{48(e-1)^2} \right] + \frac{72\kappa^3 V_{ac}}{\pi^2} \left(\frac{T}{\kappa} \right) \left[Z_4(0) - Z_4(1) - \frac{(4e-3)}{72(e-1)^2} \right] \quad (5.4)$$

Note that to get the above linear dependence of C_V on T by differentiating the high T behaviour of U we need to expand U upto T^2 order. The behaviour for equilibrium number of photons is

$$\bar{N} \approx -\frac{4\kappa^3 V_{ac}}{\pi^2} \left[Z_3(0) - Z_3(1) - \frac{1}{4(e-1)} \right] + \frac{6\kappa^3 V_{ac}}{\pi^2} \left(\frac{T}{\kappa} \right) \left[Z_3(0) - Z_3(1) - \frac{1}{6(e-1)} \right] \quad (5.5)$$

This linear behaviour for $T \rightarrow \kappa$ is very clearly visible in the plots. In the low temperature regime we take $\frac{T}{\kappa} = \epsilon < 1$. Since $Li_n(z) \rightarrow z$ as $z \rightarrow 0$, the leading order behaviour of the energy density will be (see (3.5))

$$u \approx \frac{\pi^2 \kappa^4 \epsilon^4}{15} - \frac{\kappa^2}{\pi^2} \epsilon e^{-\frac{1}{\epsilon}} \approx \frac{\pi^2 T^4}{15} \quad (5.6)$$

We have neglected the second term with respect to the first. This can be seen by putting $x = \frac{1}{\epsilon}$ and as $x \rightarrow \infty$ the ratio of second term to the first in the above equation goes to zero. For pressure we do the similar analysis where the first term in (3.18) is nothing but $\left(\frac{\kappa^3 T}{3\pi^2} \right) Li_1(e^{-\frac{\kappa}{T}})$ and we get

$$P \approx \frac{\pi^2 T^4}{45} \quad (5.7)$$

Other quantities, similarly, are as follows,

$$S \approx \frac{4V_{ac}\pi^2 T^3}{45} \quad (5.8)$$

$$C_V \approx \frac{4V_{ac}\pi^2 T^3}{15} \quad (5.9)$$

$$\bar{N} \approx \frac{2V_{ac}\zeta(3)T^3}{\pi^2} \quad (5.10)$$

Therefore in low temperature regime, energy density u and radiation pressure P follow $\sim T^4$ behaviour, while the entropy S , the specific heat C_V and the equilibrium number of photons follow $\sim T^3$ behaviour. For the modified measure we get essentially the similar behaviour. In high temperature case the behaviour is

$$u \approx u_a + u_b T \quad (5.11)$$

where u_a and u_b is

$$u_a = -\frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'} \frac{4\pi}{(3-n')} \left(\frac{\kappa^4}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+4)(n+3) \left[Z_{n+4}(0) - Z_{n+4}(1) - \frac{1}{(n+3)!(n+3)(e-1)} \right] \\ - \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi\kappa^4}{3V_{ac}\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+4)(n+3) \left[Z_{n+4}(0) - Z_{n+4}(1) - \frac{1}{(n+3)!(n+3)(e-1)} \right] \quad (5.12)$$

$$\begin{aligned}
u_b = & \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!} \frac{4\pi}{(3-n')} \left(\frac{\kappa^4}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+4) \frac{(n+4)}{\kappa} \left[Z_{n+4}(0) - Z_{n+4}(1) - \frac{1}{(n+3)!(n+4)(e-1)} \right] \\
& + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi\kappa^4}{3V_{ac}\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+4) \frac{(n+4)}{\kappa} \left[Z_{n+4}(0) - Z_{n+4}(1) - \frac{1}{(n+3)!(n+4)(e-1)} \right]
\end{aligned} \quad (5.13)$$

$$P \approx P_a + P_b T \quad (5.14)$$

where P_a and P_b is

$$\begin{aligned}
P_a = & -\frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!} \frac{4\pi}{(3-n')} \left(\frac{\kappa^4}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+4) \left[Z_{n+4}(0) - Z_{n+4}(1) \right] \\
& - \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi\kappa^4}{3V_{ac}\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+4) \left[Z_{n+4}(0) - Z_{n+4}(1) \right]
\end{aligned} \quad (5.15)$$

$$\begin{aligned}
P_b = & \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!} \frac{4\pi}{(3-n')} \left(\frac{\kappa^4}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left\{ \Gamma(n+4) \frac{(n+4)}{\kappa(n+3)} \left[Z_{n+4}(0) - Z_{n+4}(1) \right] - \frac{1}{\kappa(n+3)} \ln \left(1 - \frac{1}{e} \right) \right\} \\
& + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi\kappa^4}{3V_{ac}\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \left\{ \Gamma(n+4) \frac{(n+4)}{\kappa(n+3)} \left[Z_{n+4}(0) - Z_{n+4}(1) \right] - \frac{1}{\kappa(n+3)} \ln \left(1 - \frac{1}{e} \right) \right\}
\end{aligned} \quad (5.16)$$

$$S \approx S_a + S_b T \quad (5.17)$$

where S_a and S_b is

$$\begin{aligned}
S_a = & -\frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \frac{1}{(n+3)} \left\{ \Gamma(n+4)(n+4)(n+2) \left[Z_{n+4}(0) - Z_{n+4}(1) \right] \right. \\
& \left. - \frac{1}{(n+3)!(n+2)(e-1)} \right] + \frac{1}{(e-1)} + \ln \left(1 - \frac{1}{e} \right) \Big\} \\
& - \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \frac{1}{(n+3)} \left\{ \Gamma(n+4)(n+4)(n+2) \left[Z_{n+4}(0) - Z_{n+4}(1) \right] \right. \\
& \left. - \frac{1}{(n+3)!(n+2)(e-1)} \right] + \frac{1}{(e-1)} + \ln \left(1 - \frac{1}{e} \right) \Big\}
\end{aligned} \quad (5.18)$$

$$\begin{aligned}
S_b = & -\frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left\{ \Gamma(n+4) \frac{(n+4)}{\kappa} \left[Z_{n+4}(0) - Z_{n+4}(1) \right] \right. \\
& \left. - \frac{1}{(n+3)!(n+4)(e-1)} \right] \Big\} \\
& - \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \left\{ \Gamma(n+4) \frac{(n+4)}{\kappa} \left[Z_{n+4}(0) - Z_{n+4}(1) \right] \right. \\
& \left. - \frac{1}{(n+3)!(n+4)(e-1)} \right] \Big\}
\end{aligned} \quad (5.19)$$

$$C_V \approx C_{V_a} + C_{V_b} T \quad (5.20)$$

where C_{V_a} and C_{V_b} is

$$\begin{aligned} C_{V_a} = & -\frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left\{ \Gamma(n+4)(n+4)(n+2) \left[Z_{n+4}(0) - Z_{n+4}(1) \right. \right. \\ & \left. \left. - \frac{1}{(n+3)!(n+2)(e-1)} \right] + \frac{(e-2)}{(e-1)^2} \right\} \\ & - \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \left\{ \Gamma(n+4)(n+4)(n+2) \left[Z_{n+4}(0) - Z_{n+4}(1) \right. \right. \\ & \left. \left. - \frac{1}{(n+3)!(n+2)(e-1)} \right] + \frac{(e-2)}{(e-1)^2} \right\} \end{aligned} \quad (5.21)$$

$$\begin{aligned} C_{V_b} = & \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \left\{ \Gamma(n+4)(n+4)(n+3) \frac{1}{\kappa} \left[Z_{n+4}(0) - Z_{n+4}(1) - \frac{1}{(n+3)!(n+3)(e-1)} \right. \right. \\ & \left. \left. + \frac{1}{(n+3)!(n+4)(n+3)(e-1)^2} \right] \right\} \\ & + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \left\{ \Gamma(n+4)(n+4)(n+3) \frac{1}{\kappa} \left[Z_{n+4}(0) - Z_{n+4}(1) - \frac{1}{(n+3)!(n+3)(e-1)} \right. \right. \\ & \left. \left. + \frac{1}{(n+3)!(n+4)(n+3)(e-1)^2} \right] \right\} \end{aligned} \quad (5.22)$$

$$\bar{N} \approx \bar{N}_a + \bar{N}_b T \quad (5.23)$$

where \bar{N}_a and \bar{N}_b is

$$\begin{aligned} \bar{N}_a = & -\frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+3) \left\{ (n+2) \left[Z_{n+3}(0) - Z_{n+3}(1) - \frac{1}{(n+2)!(n+2)(e-1)} \right] \right\} \\ & - \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+3) \left\{ (n+2) \left[Z_{n+3}(0) - Z_{n+3}(1) - \frac{1}{(n+2)!(n+2)(e-1)} \right] \right\} \end{aligned} \quad (5.24)$$

$$\begin{aligned} \bar{N}_b = & \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!} \frac{4\pi}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+3) \left\{ \frac{(n+3)}{\kappa} \left[Z_{n+3}(0) - Z_{n+3}(1) - \frac{1}{(n+2)!(n+3)(e-1)} \right] \right\} \\ & + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!} \left(\frac{4\pi}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+3) \left\{ \frac{(n+3)}{\kappa} \left[Z_{n+3}(0) - Z_{n+3}(1) - \frac{1}{(n+2)!(n+3)(e-1)} \right] \right\} \end{aligned} \quad (5.25)$$

The low temperature limit can be calculated as we did in case of unmodified measure to get

$$\begin{aligned} u \approx & \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{T^{n+4}}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+4) Z_{n+4}(0) \\ & + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi T^{n+4}}{3V_{ac}\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+4) Z_{n+4}(0) \end{aligned} \quad (5.26)$$

$$\begin{aligned}
P \approx & \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{T^{n+4}}{V_{ac}\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \frac{\Gamma(n+4)}{(n+3)} Z_{n+4}(0) \\
& + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi T^{n+4}}{3V_{ac}\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \frac{\Gamma(n+4)}{(n+3)} Z_{n+4}(0)
\end{aligned} \tag{5.27}$$

$$\begin{aligned}
S \approx & \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{T^{n+3}}{\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \frac{\Gamma(n+4)(n+4)}{(n+3)} Z_{n+4}(0) \\
& + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi T^{n+3}}{3\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \frac{\Gamma(n+4)(n+4)}{(n+3)} Z_{n+4}(0)
\end{aligned} \tag{5.28}$$

$$\begin{aligned}
C_V \approx & \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^n} \frac{4\pi}{(3-n')} \left(\frac{T^{n+3}}{\kappa^3} \right) \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+4)(n+4) Z_{n+4}(0) \\
& + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^n} \left(\frac{4\pi T^{n+3}}{3\kappa^3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+4)(n+4) Z_{n+4}(0)
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
\bar{N} \approx & \frac{1}{(\pi)^2} \sum_{\substack{n=0, n'=0 \\ n' \neq 3}}^{\infty} \frac{a_{n,n'}}{n!n'!\kappa^{n+3}} \frac{4\pi T^{n+3}}{(3-n')} \left[\left(\frac{3V\kappa^3}{4\pi} \right)^{\frac{3-n'}{3}} - 1 \right] \Gamma(n+3) Z_{n+3}(0) \\
& + \frac{1}{(\pi)^2} \sum_{n=0}^{\infty} \frac{a_{n,3}}{n!3!\kappa^{n+3}} \left(\frac{4\pi T^{n+3}}{3} \right) \ln \left(\frac{3\kappa^3 V}{4\pi} \right) \Gamma(n+3) Z_{n+3}(0)
\end{aligned} \tag{5.30}$$

6 Summary

We started with DSR formalism by MS where the modified dispersion relation is given by (1.1). But in case of photon gas it is simply $\varepsilon = p$. Since there is a cut-off on maximum energy and minimum length, the expression of the thermodynamic quantities change accordingly. We started by considering the change in the phase space measure for exotic spacetimes and discussed the example of classical ideal gas for illustration. We then considered a model of a photon gas obeying Bose-Einstein statistics in grand canonical ensemble and went on calculating various thermodynamic quantities such as energy density, pressure, entropy, specific heat and equilibrium number of photons. We found a one to one correspondence between the DSR photons and the acoustic phonons in the Debye theory. The Stefan-Boltzmann law got modified which may give correction to the dynamics of many stellar objects. We then calculated the possible change in the thermodynamic quantities due to the change in the phase space measure. Because of this modification, Planck's energy density distribution and the Wien's displacement law got modified. We have plotted various thermodynamic quantities. The leading behaviour for $T \rightarrow \kappa$ and $T \rightarrow 0$ have also been calculated.

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Appendices

A Criterion for swaping the double summation and the integral

We consider the following equality,

$$\int_{\mathcal{M}} \sum_n \sum_{n'} A_{n,n'} = \sum_n \sum_{n'} \int_{\mathcal{M}} A_{n,n'} \quad (\text{A.1})$$

In this appendix we will show that the above equality holds if,

$$\sum_n \sum_{n'} \int_{\mathcal{M}} |A_{n,n'}| < \infty \quad (\text{A.2})$$

We take the two summations out of the integral one by one in the LHS of (A.1) to get the RHS, using theorem 1.38 of [13] which is allowed if,

$$\sum_n \int_{\mathcal{M}} \left| \sum_{n'} A_{n,n'} \right| < \infty \quad (\text{A.3})$$

and

$$\sum_{n'} \int_{\mathcal{M}} |A_{n,n'}| < \infty \quad \forall n \quad (\text{A.4})$$

Let us rewrite A.2 as,

$$\sum_n t_n < \infty \quad (\text{A.5})$$

where,

$$t_n = \sum_{n'} \int_{\mathcal{M}} |A_{n,n'}| \quad \forall n \quad (\text{A.6})$$

We note that $t_n \geq 0 \quad \forall n$, which along with (A.5) implies $t_n < \infty \quad \forall n$ which is nothing but (A.4). This further implies (because of theorem 1.38 of [13]),

$$t_n = \int_{\mathcal{M}} \sum_{n'} |A_{n,n'}| \quad \forall n \quad (\text{A.7})$$

Now we note that,

$$\left| \sum_{n'} A_{n,n'} \right| \leq \sum_{n'} |A_{n,n'}| \quad \forall n \quad (\text{A.8})$$

Integrating over \mathcal{M} followed by the summation over n we get,

$$\sum_n \int_{\mathcal{M}} \left| \sum_{n'} A_{n,n'} \right| \leq \sum_n \int_{\mathcal{M}} \sum_{n'} |A_{n,n'}| \quad \forall n \quad (\text{A.9})$$

Taking A.7 and (A.5) into account in the above inequality we get (A.3). Thus we have proved that (A.2) implies (A.3) and (A.4) and hence if (A.2) is satisfied then the equality (A.1) holds true.

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